Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(02)00025-4

J. Appl. Maths Mechs, Vol. 66, No. 2, pp. 201–213, 2002

0021-8928/02/$—see front matter

ASYMPTOTIC EXPANSION OF SOLUTIONS OF OPTIMAL CONTROL PROBLEMS FOR DISCRETE WEAKLY CONTROLLABLE SYSTEMS†

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(*Received* 22 *February* 2001)

Asymptotic expansions of solutions of optimal control problems for weakly controllable systems are constructed as series of non-negative integral powers of a small parameter. Estimates are obtained for the closeness of the approximate solution to the exact solution, and it is shown that the values of the minimized functional do not increase when higher-order approximations to the control are used. © 2002 Elsevier Science Ltd. All rights reserved.

Control by small signals is encountered in controlling spacecraft with low thrust (electronuclear engines, a solar sail, etc.), in a variety of correction problems and in economics [1]. A first approximation to the solution has been constructed in the problem of continuous optimal control of weakly controllable systems without restrictions on the values of the controls, by substituting a postulated asymptotic expansion of the solution into the condition of the problem [1]. Continuous weakly controllable systems with a restriction on the control have been considered within the limits of the first approximation [2–6]. The method of constructing an asymptotic expansion of the solution of an optimal control problem by substituting the postulated asymptotic expansion into the condition of the problem, and then determining a series of optimal control problems to find the expansion terms, have been used for singularly perturbed systems without restrictions on the values of the controls; this method has been called the "direct scheme" [7, 8]. In that context, asymptotic expansions of the solution, of arbitrary accuracy, have been constructed; estimates have been obtained for the closeness of the approximate solution to the exact one, and it has been established that the value of the minimized functional does not increase with each new approximation [7, 8]. Similar results have been obtained for continuous optimal control problems for weakly controllable systems [9], and for non-linear discrete optimal control problems with small step-size, without restrictions on the control [10, 11].

In this paper, the direct scheme is used to construct asymptotic expansions of solutions of optimal control problems for discrete weakly controllable systems without restrictions on the control. Estimates are proved for the closeness of the approximate solution to the exact one, in terms of the control, the trajectory and the functional, and it is shown that the values of the minimized functional do not increase when a new approximation to the control is used.

1. FORMULATION OF THE PROBLEM

The following discrete problem is considered for a weakly controllable system

$$P_\varepsilon : J_\varepsilon(u) = \sum_{k=0}^{N} F_k(x(k)) + \varepsilon \sum_{l=0}^{N-l} G_l(x(l), u(l)) \to \min_u \tag{1.1}$$

$$x(l+1) = f_l(x(l)) + \varepsilon q_l(x(l), u(l)), \quad l = 0, 1, \ldots, N-1 \tag{1.2}$$

$$x(0) = x^0 \tag{1.3}$$

where

$$x(k) \in X, \quad k = 0, 1, \ldots, N, \quad u(l) \in U, \quad l = 0, 1, \ldots, N-1$$

†*Prikl. Mat. Mekh.* Vol. 66, No. 2, pp. 214–227, 2002.

X and U are real finite-dimensional Euclidean spaces and the number of steps N is fixed; everywhere henceforth, it will be assumed that k takes the values 0, 1, ..., N, l takes the values 0, 1, ..., N–1, F_k and G_l are scalar functions, f_l and g_l are functions with values in X, and $\varepsilon > 0$ is a small parameter; the functions F_k, G_l, f_l and g_l occurring in (1.1) and (1.2) are assumed to be continuously differentiable a sufficient number of times with respect to their arguments.

Where $\varepsilon = 0$ system (1.2) turns out to be uncontrollable and, solving the problem

$$P_0 : x(l+1) = f_l(x(l)), x(0) = x^0$$

one obtains the unperturbed trajectory.

Asymptotic expansions of the solution of problem (1.1)–(1.3) will be constructed using the direct scheme. The results of this paper were announced by the author in [12].

2. THE FORMALISM OF THE CONSTRUCTION OF ASYMPTOTIC EXPANSIONS

A solution of the perturbed problem (1.1)–(1.3) will the sought in series form

$$x(k) = \sum_{j\geqslant 0} \varepsilon^j x_j(k), u(l) = \sum_{j\geqslant 0} \varepsilon^j u_j(l) \tag{2.1}$$

We substitute relations (2.1) into expressions (1.1)–(1.3), expand the right-hand sides of (1.1) and (1.2) in series in powers of ε, and then equate the coefficients of like powers of ε in Eqs (1.2) and (1.3). Then the functional to be minimized may be written in the form

$$J_\varepsilon(u) = \sum_{j\geqslant 0} \varepsilon^j J_j \tag{2.2}$$

and relations (1.2) and (1.3) yield equations and initial conditions for the quantities $x_j(k)$

$$\begin{aligned} &x_0(l+1) = \bar{f}_l, x_1(l+1) = (\bar{f}_l)_x x_1(l) + \bar{g}_l, \dots \\ &x_j(l+1) = (\bar{f}_l)_x x_j(l) + (\bar{g}_l)_u u_{j-1}(l) + [\tilde{f}_l + \varepsilon\tilde{g}_l]_j, \quad j > 1 \end{aligned} \tag{2.3}$$

$$x_0(0) = x^0, \quad x_j(0) = 0, \quad j \geqslant 1 \tag{2.4}$$

Throughout this paper, and expression of the form $(\bar{f}_l)_x$ will stand for the derivative of the function f_l with respect to the variable x and the bar will mean that the values of the appropriate functions and their derivatives are calculated at $x = x_0$, $u = u_0$. The following notation will be used for the expansion of an arbitrary function $h = h(\varepsilon)$ in powers of ε

$$h(\varepsilon) = \sum_{j\geqslant 0} \varepsilon^j h_j = \{h\}_{n-1} + \varepsilon^n [h]_n + \alpha(\varepsilon^{n+1}), \ \{h\}_{n-1} = \sum_{j=0}^{n-1} \varepsilon^j h_j, \ [h]_n = h_n$$

where $\alpha(\varepsilon^{n+1})$ denotes the sum of terms of the expansion of order ε^{n+1} and higher. A tilde over a function will mean that the function is evaluated at $x(k) = \tilde{x}_{j-1}(k)$, $u(l) = \tilde{u}_{j-2}(l)$, where

$$\tilde{x}_{j-1}(k) = \sum_{i=0}^{j-1} \varepsilon^i x_i(k), \ \tilde{u}_{j-2}(l) = \sum_{i=0}^{j-2} \varepsilon^i u_i(l) \tag{2.5}$$

In particular, we have

$$[\tilde{f}_l + \varepsilon\tilde{g}_l]_2 = \frac{1}{2}(\bar{f}_l)_{x^2} x_1^2(l) + (\bar{g}_l)_x x_1(l)$$

$$[\tilde{f}_l + \varepsilon\tilde{g}_l]_3 = (\bar{f}_l)_{x^2} x_1(l) x_2(l) + \frac{1}{3!}(\bar{f}_l)_{x^3} x_1^3(l) +$$

$$+ (\bar{g}_l)_x x_2(l) + \frac{1}{2}((\bar{g}_l)_{x^2} x_1^2(l) + (\bar{g}_l)_{u^2} u_1^2(l)) + (\bar{g}_l)_{xu} u_1(l) x_1(l)$$

It is obvious that $x_0(k)$ is a solution of problem P_0.

Let us write down the first coefficients of expansion (2.2). The coefficient $J_0 = \sum_k \bar{F}_k$ is known after Problem P_0 has been solved. We have

$$J_1 = \sum_k (\bar{F}_k)_x x_1(k) + \sum_l \bar{G}_l$$

$$J_2 = \sum_k ((\bar{F}_k)_x x_2(k) + \frac{1}{2}(\bar{F}_k)_{x^2} x_1^2(k)) + \sum_l ((\bar{G}_l)_x x_1(l) + (\bar{G}_l)_u u_1(l))$$

$$J_3 = \sum_k ((\bar{F}_k)_x x_3(k) + (\bar{F}_k)_{x^2} x_1(k) x_2(k) + \frac{1}{3!}(\bar{F}_k)_{x^3} x_1^3(k)) +$$

$$+ \sum_l ((\bar{G}_l)_x x_2(l) + (\bar{G}_l)_u u_2(l) + \frac{1}{2}((\bar{G}_l)_{x^2} x_1^2(l) + (\bar{G}_l)_{u^2} u_1^2(l)) + (\bar{G}_l)_{xu} u_1(l) x_1(l))$$

To determine the pair of functions $(x_1(k)_l\, u_0(l))$, we consider the problem

$$P_1 : \tilde{J}_1(u_0) = J_1(u_0) = (\bar{F}_N)_x x_1(N) + \sum_l ((\bar{F}_l)_x x_1(l) + \bar{G}_l) \to \min_{u_0}$$

$$x_1(l+1) = (\bar{f}_l)_x x_1(l) + \bar{g}_l, x_1(0) = 0$$

Note that in Problem P_1 the functional to be minimized and the equation of state are linear with respect to $x_1(l)$.

We shall assume that the following condition is satisfied.

Condition 1. Problem P_1 has a unique solution which is defined by the following equality (see, e.g. [13])

$$(H_1(l))_{u_0(l)} = -(\bar{G}_l)_u + \psi_0(l+1)(\bar{g}_l)_u = 0 \tag{2.6}$$

The Hamiltonian $H_1(l)$ is given by the formula

$$H_1(l) = -(\bar{F}_l)_x x_1(l) - \bar{G}_l + \psi_0(l+1)((\bar{f}_l)_x x_1(l) + \bar{g}_l)$$

The adjoint variable ψ_0 is the solution of the problem

$$\psi_0(l) = -(\bar{F}_l)_x + \psi_0(l+1)(\bar{f}_l)_x, \quad \psi_0(N) = -(\bar{F}_N)_x \tag{2.7}$$

Problem (2.7), which determines the adjoint variable, is solved independently of the problem of determining the state $x_1(k)$ and the control $u_0(l)$. If Eq. (2.6) yields a unique expression for $u_0(l)$ in terms of $\psi_0(l+1)$, then, after finding the adjoint variable and then the control, the state $x_1(k)$ is found from the recurrence relations (2.3) of the equation of state.

Using relations (2.7), (2.6), (2.3) and (2.4), let us transform the terms in the expression for J_2 which remain unknown after Problems P_0 and P_1 have been solved. We obtain

$$\sum_k (\bar{F}_k)_x x_2(k) + \sum_l (\bar{G}_l)_u u_1(l) =$$

$$= \sum_l (-\psi_0(l) + \psi_0(l+1)(\bar{f}_l)_x) x_2(l) - \psi_0(N) x_2(N) + \sum_l \psi_0(l+1)(\bar{g}_l)_u u_1(l) =$$

$$= -\sum_k \psi_0(k) x_2(k) + \sum_l \psi_0(l+1)(x_2(l+1) - \frac{1}{2}(\bar{f}_l)_{x^2} x_1^2(l) - (\bar{g}_l)_x x_1(l)) =$$

$$= -\sum_l \psi_0(l+1)\left(\frac{1}{2}(\bar{f}_l)_{x^2} x_1^2(l) + (\bar{g}_l)_x x_1(l)\right)$$

Thus, the coefficient J_2 in expansion (2.2) depends only on the solutions of Problems P_0 and P_1.

Carrying out similar transformations for part of the terms in the expression for J_3, we have

$$\sum_k (\bar{F}_k)_x x_3(k) + \sum_l (\bar{G}_l)_u u_2(l) = -\sum_l \psi_0(l+1)((\bar{f}_l)_{x^2} x_1(l) x_2(l) + \frac{1}{3!}(\bar{f}_l)_{x^3} x_1^3(l) +$$

$$+ (\bar{g}_l)_x x_2(l) + \frac{1}{2}((\bar{g}_l)_{x^2} x_1^2(l) + (\bar{g}_l)_{u^2} u_1^2(l)) + (\bar{g}_l)_{xu} u_1(l) x_1(l))$$

Taking the last relation into account, let $\tilde{J}_2$ denote the sum of those terms in the expression for J_3 that are still unknown after Problems P_0 and P_1 have been solved.

To determine the pair of functions (x_2, u_1), consider the linearly quadratic problem

$$P_2 : \tilde{J}_2(u_1) = Q_N x_2(N) + \sum_l \left(Q_l x_2(l) + \frac{1}{2} R_l u_1^2(l) + S_l u_1(l) \right) \to \min_{u_1}$$

$$x_2(l+1) = (\bar{f}_l)_x x_2(l) + (\bar{g}_l)_u u_1(l) + [\tilde{f}_l + \varepsilon \tilde{g}_l]_2, \quad x_2(0) = 0$$

The functions $Q_N = (\bar{F}_N)_{x^2} x_1(N)$, $Q_l = (\bar{F}_l)_{x^2} x_1(l) + (\bar{G}_l)_x - \psi_0(l+1)((\bar{f}_l)_{x^2} x_1(l) + (\bar{g}_l)_x)$, $R_l = (\bar{G}_l)_{u^2} - \psi_0(l+1)((\bar{g}_l)_{u^2}$, $S_l = ((\bar{G}_l)_{ux} - \psi_0(l+1)(\bar{g}_l)_{ux}) x_1(l)$ depend on the solutions of Problems P_0 and P_1.

We shall assume that the following condition is satisfied.

Condition 2. The operators R_l are positive definite.

Then the linearly quadratic problem P_2 is uniquely solvable, that is, the second approximation (x_2, u_1) is uniquely defined.

Let us write down the Hamiltonian for Problem P_ε (see, e.g. [13])

$$H(l) = -F_l(x(l)) - \varepsilon G_l(x(l), u(l)) + \psi(l+1)(f_l(x(l)) + \varepsilon g_l(x(l), u)(l)))$$

where the adjoint variable ψ is the solution of the problem

$$\psi(l) = (H(l))_{x(l)} = -(F_l(x(l)) + \varepsilon G_l(x(l), u(l)))_x + \\ + \psi(l+1)(f_l(x(l)) + \varepsilon g_l(x(l), u(l)))_x \tag{2.8}$$

$$\psi(N) = -(F_N(x(N)))_x \tag{2.9}$$

Since the problem under consideration involves no restrictions on the control, a necessary condition for an optimal control in Problem P_ε is that

$$(H(l))_{u(l)} = \varepsilon(-(G_l(x(l), u(l)))_u + \psi(l+1)(g_l(x(l), u(l)))_u) = 0 \tag{2.10}$$

Let us substitute expansions (2.1) and the expression

$$\psi(k) = \sum_{j \geq 0} \varepsilon^j \psi_j(k) \tag{2.11}$$

into relations (2.8)–(2.10).

Equating coefficients of the least and next powers of ε in the expressions obtained, we obtain Eqs (2.6) and (2.7), which follow from the optimality condition for the control in Problem P_1, and the equalities

$$\psi_1(l) = -(\bar{F}_l)_{x^2} x_1(l) - (\bar{G}_l)_x + \psi_0(l+1)((\bar{f}_l)_{x^2} x_1(l) + (\bar{g}_l)_x) + \psi_1(l+1)(\bar{f}_l)_x \tag{2.12}$$

$$\psi_1(N) = -(\bar{F}_N)_{x^2} x_1(N) \tag{2.13}$$

$$-(\bar{G}_l)_{ux} x_1(l) - (\bar{G}_l)_{u^2} u_1(l) + \psi_0(l+1)((\bar{g}_l)_{ux} x_1(l) + (\bar{g}_l)_{u^2} u_1(l)) + \\ + \psi_1(l+1)(\bar{g}_l)_u = 0 \tag{2.14}$$

If we write down the Maximum Principle for Problem P_2 (see, e.g. [13]), the control $u_1(l)$ will satisfy Eq. (2.14), and the adjoint variable $\psi_1(k)$ will be a solution of problem (2.12)–(2.13).

Thus, the equations for the state, control and adjoint variable, obtained using the necessary condition for a control in Problems P_1 and P_2 to be optimal, are identical with the equations for the corresponding approximation of the asymptotic expansion of the solution of problem (1.2), (1.3), (2.8)–(2.10), obtained from the necessary condition for a control in Problem P_ε to be optimal.

We now proceed to higher approximations.

Consider Problems $P_j (j \geqslant 0)$. For $j = 0, 1$, Problems P_0 and P_1 have already been defined; for $j \geqslant 2$, Problems P_j are linear-quadratic problems of the following form

$$P_j : \bar{J}_j(u_{j-1}) = [(\tilde{F}_N)_x]_{j-1} x_j(N) + \sum_l ([-\tilde{\psi}_{j-2}(l+1)(\tilde{f}_l + \varepsilon\tilde{g}_l)_x +$$

$$+ (\tilde{F}_l + \varepsilon\tilde{G}_l)_x]_{j-1} x_j(l) + \frac{1}{2} R_l u_{j-1}^2(l) + [-\tilde{\psi}_{j-2}(l+1)(\tilde{g}_l)_u + (\tilde{G}_l)_u]_{j-1} u_{j-1}(l)) \to \min_{u_{j-1}} \tag{2.15}$$

$$x_j(l+1) = (\bar{f}_l)_x x_j(l) + (\bar{g}_l)_u u_{j-1}(l) + [\tilde{f}_l + \varepsilon\tilde{g}_l]_j, \quad x_j(0) = 0 \tag{2.16}$$

Recall that a tilde over the symbols for the functions f_l, g_l, F_k, G_l and their derivatives means that they are evaluated at $x = \tilde{x}_{j-1}$, $u = \tilde{u}_{j-2}$, where $\tilde{x}_{j-1}(k)$ and $\tilde{u}_{j-2}(l)$ are defined by (2.5), the pair (x_i, u_{i-1}) is a solution of Problem P_i, and the function $\tilde{\psi}_{j-2}$ is defined by the equality

$$\tilde{\psi}_{j-2}(k) = \sum_{i=0}^{j-2} \varepsilon^i \psi_i(k) \tag{2.17}$$

where $\psi_i(k)$ is the adjoint variable in Problem P_{i+1}.

Note that the equation of state in Problem P_j is the coefficient of ε^j in the equality obtained by substituting (2.1) into (1.2) and expanding the resulting expressions in series in powers of ε.

For $j = 2$, relations (2.15) and (2.16) yield Problem P_2, already considered, in which the performance criterion $\bar{J}_2(u_1)$ is the transformed expression of the coefficient J_3 in expansion (2.2), omitting the terms known after Problems P_0 and P_1 have been solved.

The Hamiltonian for Problem P_j ($j \geqslant 2$) is

$$H_j(l) = -[-\tilde{\psi}_{j-2}(l+1)(\tilde{f}_l + \varepsilon\tilde{g}_l)_x + (\tilde{F}_l + \varepsilon\tilde{G}_l)_x]_{j-1} x_j(l) -$$

$$- \tfrac{1}{2} R_l u_{j-1}^2(l) - [-\tilde{\psi}_{j-2}(l+1)(\tilde{g}_l)_u + (\tilde{G}_l)_u]_{j-1} u_{j-1}(l) +$$

$$+ \psi_{j-1}(l+1)((\bar{f}_l)_x x_j(l) + (\bar{g}_l)_u u_{j-1}(l) + [\tilde{f}_l + \varepsilon\tilde{g}_l]_j)$$

where the adjoint variable ψ_{j-1} is the solution of the problem

$$\psi_{j-1}(l) = (H_j(l))_{x_j(l)} = -[-\tilde{\psi}_{j-2}(l+1)(\tilde{f}_l + \varepsilon\tilde{g}_l)_x + (\tilde{F}_l + \varepsilon\tilde{G}_l)_x]_{j-1} +$$
$$+ \psi_{j-1}(l+1)(\bar{f}_l)_x \tag{2.18}$$

$$\psi_{j-1}(N) = -[(\tilde{F}_N)_x]_{j-1} \tag{2.19}$$

The necessary condition for optimality of the control in Problem P_j ($j \geqslant 2$) is the equality

$$(H_j(l))_{u_{j-1}(l)} = -R_l u_{j-1}(l) - [-\tilde{\psi}_{j-2}(l+1)(\tilde{g}_l)_u + (\tilde{G}_l)_u]_{j-1} +$$
$$+ \psi_{j-1}(l+1)(\bar{g}_l)_u = 0 \tag{2.20}$$

Theorem 1. The equations for the state, control and adjoint variable, obtained from the necessary condition for optimality of the control in Problem $P_m (m \geqslant 1)$, are identical with the equations for x_m, u_{m-1}, ψ_{m-1} from asymptotic expansions (2.1) and (2.11) of the solution of problem (1.2), (1.3), (2.8)–(2.10), obtained using the necessary condition for optimality of the control in Problem P_ε.

Proof. For $m = 1, 2$, the statement of the theorem has already been proved. Suppose it is true for $m < j$. For $j \geqslant 2$ we introduce the notation

$$\Delta x(k) = x(k) - \tilde{x}_{j-1}(k) = \varepsilon^j x_j(k) + \alpha(\varepsilon^{j+1})$$

$$\Delta\psi(k) = \psi(k) - \tilde{\psi}_{j-2}(k) = \varepsilon^{j-1}\psi_{j-1}(k) + \alpha(\varepsilon^j)$$

$$\Delta u(l) = u(l) - \tilde{u}_{j-2}(l) = \varepsilon^{j-1}u_{j-1}(l) + \alpha(\varepsilon^j) \tag{2.21}$$

where $\tilde{x}_{j-1}(k)$, $\tilde{u}_{j-2}(l)$, $\tilde{\psi}_{j-2}(k)$ are given by formulae (2.5) and (2.17),

Replacing x, u and ψ in (1.2), (1.3) and (2.8)–(2.10) by their representations in (2.21), and transforming, we obtain

$$\tilde{x}_{j-1}(l+1) + \Delta x(l+1) = \tilde{f}_l + \varepsilon\tilde{g}_l + (\tilde{f}_l + \varepsilon\tilde{g}_l)_x\Delta x(l) + \varepsilon(\tilde{g}_l)_u\Delta u(l) +$$
$$+\frac{1}{2}\varepsilon^{2j-1}(\overline{g}_l)_{u^2}u_{j-1}^2(l) + \alpha(\varepsilon^{2j}),\ \ \Delta x(0) = 0 \tag{2.22}$$

$$\tilde{\psi}_{j-2}(l) + \varepsilon^{j-1}\psi_{j-1}(l) = -(\tilde{F}_l + \varepsilon\tilde{G}_l)_x + \tilde{\psi}_{j-2}(l+1)(\tilde{f}_l + \varepsilon\tilde{g}_l)_x +$$
$$+\ \varepsilon^{j-1}\psi_{j-1}(l+1)(\tilde{f}_l)_x + \alpha(\varepsilon^j) \tag{2.23}$$

$$\tilde{\psi}_{j-2}(N) + \varepsilon^{j-1}\psi_{j-1}(N) = -(\tilde{F}_N)_x + \alpha(\varepsilon^j)$$

$$-(\tilde{G}_l)_u - \varepsilon^{j-1}(\overline{G}_l)_{u^2}u_{j-1}(l) + \tilde{\psi}_{j-2}(l+1)(\tilde{g}_l)_u +$$
$$+\ \varepsilon^{j-1}\psi_0(l+1)(\overline{g}_l)_{u^2}u_{j-1}(l) + \varepsilon^{j-1}\psi_{j-1}(l+1)(\overline{g}_l)_u = \alpha(\varepsilon^j) \tag{2.24}$$

Equating the coefficients of ε^j in (2.2) and the coefficients of ε^{j-1} in (2.23) and (2.24), we obtain relations (2.16) and (2.18)–(2.20), which follow from the necessary condition for optimality of the control in Problem P_j. This establishes the statement of the theorem for $m = j$, and thereby proves Theorem 1.

Assuming that Conditions 1 and 2 are satisfied the following theorem holds.

Theorem 2. The coefficient J_{2m} in expansion (2.2) is known after Problems P_i $(i = 0, 1, \dots, m)$ have been solved, from which one finds $x_i(k)$, $u_{i-1}(l)$ $(i \geqslant 1)$. The transformed expression for the coefficient J_{2m+1}, omitting terms known after Problems P_i $(i = 0, 1, \dots, m)$ have been solved, is identical with the performance criterion $\tilde{J}_{m+1}$ in Problem P_{m+1}.

Proof. If $m = 0$, the coefficient $J_{2m} = J_0$ is known after Problem P_0 has been solved, and the coefficient J_{2m+1} is the performance criterion in Problem P_1.

If $m = 1$, it has already been proved that $J_{2m} = J_2$ is known after Problems P_0 and P_1 have been solved, and the transformed expression for J_3 yields the performance criterion $\tilde{J}_2$ in the linearly quadratic Problem P_2; by Condition 2, this defines (x_2, u_1) uniquely.

The theorem is thus true for $m = 0, 1$.

Now suppose the statement of the theorem is true for $0 \leqslant m \leqslant n-1$ $(n \geqslant 2)$.

Let us assume that solutions have been found in Problems $P_j (j = 0, 1, \dots, n)$. Then $\tilde{x}_n(k)$, $\tilde{u}_{n-1}(l)$, $\tilde{\psi}_{n-1}(k)$, as defined by formulae (2.5) and (2.17) with $j = n + 1$, are known functions.

Let us transform the expression for $J_\varepsilon(u)$ from (1.1), replacing x and u by their representations according to (2.21) with $j = n + 1$. This gives

$$J_\varepsilon(u) = \sum_k\ (\tilde{F}_k + (\tilde{F}_k)_x\Delta x(k)) + \varepsilon\sum_l\ (\tilde{G}_l + (\tilde{G}_l)_x\Delta x(l) +$$
$$+\ (\tilde{G}_l)_u\Delta u(l) + \frac{\varepsilon^{2n}}{2}(\overline{G}_l)_{u^2}u_n^2(l)) + \alpha(\varepsilon^{2n+2}) \tag{2.25}$$

where the tilde over the symbols for the functions and their derivatives means that they are evaluated at $x(k) = \tilde{x}_n(k)$, $u(l) = \tilde{u}_{n-1}(l)$.

Using the notation introduced previously, we deduce from (2.25) and (2.23), (2.24) with $j = n + 1$ that

$$J_\varepsilon(u)=\{\tilde{F}_N\}_{2n+1}+\{(\tilde{F}_N)_x\}_{n-1}\Delta x(N)+\varepsilon^{2n+1}[(\tilde{F}_N)_x]_n x_{n+1}(N)+$$
$$+\sum_l(\{\tilde{F}_l+\varepsilon\tilde{G}_l\}_{2n+1}+\{(\tilde{F}_l+\varepsilon\tilde{G}_l)_x\}_{n-1}\Delta x(l)+\varepsilon^{2n+1}[(\tilde{F}_l+\varepsilon\tilde{G}_l)_x]_n x_{n+1}(l)+$$
$$+\ \{\varepsilon(\tilde{G}_l)_u\}_n\Delta u(l)+\varepsilon^{2n+1}[(\tilde{G}_l)_u]_n u_n(l)+\frac{\varepsilon^{2n+1}}{2}(\bar{G}_l)_{u^2}u_n^2(l))+\alpha(\varepsilon^{2n+2}) \tag{2.26}$$

$$\{(\tilde{F}_l+\varepsilon\tilde{G}_l)_x\}_{n-1}=-\tilde{\psi}_{n-1}(l)+\{\tilde{\psi}_{n-1}(l+1)(\tilde{f}_l+\varepsilon\tilde{g}_l)_x\}_{n-1}$$
$$\tilde{\psi}_{n-1}(N)=-\{(\tilde{F}_N)_x\}_{n-1}$$
$$\{\varepsilon(\tilde{G}_l)_u\}_n=\{\varepsilon\tilde{\psi}_{n-1}(l+1)(\tilde{g}_l)_u\}_n$$

Taking the last three equalities into consideration, as well as the condition $\Delta x(0) = 0$ and the relation obtained from (2.22) with $j = n + 1$ by multiplying it by $\tilde{\psi}_{n-1}(l + 1)$

$$\{\tilde{\psi}_{n-1}(l+1)(\tilde{f}_l+\varepsilon\tilde{g}_l)_x\}_{n-1}\Delta x(l)+\{\varepsilon\tilde{\psi}_{n-1}(l+1)(\tilde{g}_l)_u\}_n\Delta u(l)=$$
$$=\{\tilde{\psi}_{n-1}(l+1)\Delta x(l+1)\}_{2n+1}+\{\tilde{\psi}_{n-1}(l+1)\tilde{x}_n(l+1)-$$
$$-\tilde{\psi}_{n-1}(l+1)(\tilde{f}_l+\varepsilon\tilde{g}_l)\}_{2n+1}-\varepsilon^{2n+1}[\tilde{\psi}_{n-1}(l+1)(\tilde{f}_l+\varepsilon\tilde{g}_l)_x]_n x_{n+1}(l)-$$
$$-\varepsilon^{2n+1}[\tilde{\psi}_{n-1}(l+1)(\tilde{g}_l)_u]_n u_n(l)-\frac{\varepsilon^{2n+1}}{2}\psi_0(l+1)(\bar{g}_l)_{u^2}u_n^2(l)+\alpha(\varepsilon^{2n+2})$$

we deduce from (2.26) that

$$J_\varepsilon(u)=\{\tilde{F}_N+\sum_l(\tilde{F}_l+\varepsilon\tilde{G}_l+\tilde{\psi}_{n-1}(l+1)(\tilde{x}_n(l+1)-\tilde{f}_l-\varepsilon\tilde{g}_l)\}_{2n+1}+$$
$$+\varepsilon^{2n+1}([(\tilde{F}_N)_x]_n x_{n+1}(N)+\sum_l([-\tilde{\psi}_{n-1}(l+1)(\tilde{f}_l+\varepsilon\tilde{g}_l)_x+$$
$$+(\tilde{F}_l+\varepsilon\tilde{G}_l)_x]_n x_{n+1}(l)+[-\tilde{\psi}_{n-1}(l+1)(\tilde{g}_l)_u+(\tilde{G}_l)_u]_n u_n(l)+$$
$$+\frac{1}{2}(-\psi_0(l+1)(\bar{g}_l)_{u^2}+(\overline{G}_l)_{u^2})u_n^2(l)))+\alpha(\varepsilon^{2n+2})$$

It is obvious from this expression that J_{2n} (the coefficient of ε^{2n} in the expansion of $J_\varepsilon(u)$ in powers of ε) is known after Problems $P_j (j = 0, 1, \dots, n)$ have been solved. If we take the sum of the terms in J_{2n+1} (the coefficient of ε^{2n+1}), which depend on the unknowns $x_{n+1}(k)$ and $u_n(l)$, it is identical with the performance criterion $\bar{J}_{n+1}(u_n)$ in Problem P_{n+1} (see (2.15) with $j = n + 1$).

This completes the proof of Theorem 2.

Remarks 1. To obtain Problem P_{n+1}, it is sufficient to require that the functions occurring in (1.1) and (1.2) should possess continuous derivatives with respect to (x, u) in the neighbourhood of (x_0, u_0), of order up to and including $2n + 2$, but the smoothness of g_l and G_l may be one order lower.

2. The solution of the linearly quadratic Problem $P_j, j \geqslant 2$, assuming Condition 2 is satisfied is uniquely defined by the recurrence relations. Indeed, relations (2.18) and (2.19) uniquely define $\psi_{j-1}(l)$. Given Condition 2, $u_{j-1}(l)$ is uniquely defined from (2.20). Substituting the value found for the optimal control into the first relation of (2.16), one obtains recurrence relations (2.16), from which the optimal trajectory $x_j(k)$ can be determined.

3. ESTIMATES OF THE APPROXIMATE SOLUTION

Let us assume that solutions have been found for problems $P_j (j = 0, 1, \dots, n, n \geqslant 1)$: the functions $x_j(k)$ and $u_{j-1}(l)$ $(j \geqslant 1)$. We shall estimate the approximate solution of problem $P_\varepsilon : \tilde{x}_n(k), \tilde{u}_{n-1}(k)$ (see (2.5) with $j = n + 1$).

Lemma 1. If for the equation

$$Az=F(z,\varepsilon),\quad z\in Z \tag{3.1}$$

where Z is a Banach space, A is a linear operator defined in the space, and $\varepsilon > 0$ is a small parameter, the following conditions are satisfied:

1) A has a bounded inverse;

2) for any $\mu > 0$ constants $\delta = \delta(\mu)$ and $\varepsilon_0 = \varepsilon_0(\mu)$ exist such that, if $\|z_i\| \leq \delta(\mu)$ $(i = 1, 2)$, $0 < \varepsilon \leq \varepsilon_0$, then

$$\|F(z_1,\varepsilon) - F(z_2,\varepsilon)\| \leq \mu\|z_1 - z_2\| \tag{3.2}$$

3) for $0 < \varepsilon \leq \varepsilon_0$,

$$\|F(0,\varepsilon)\| \leq d\varepsilon^q, \quad q > 0 \tag{3.3}$$

where the constant d is independent of ε.

Then ε_1, $\mu > 0$ exist such that, for $0 < \varepsilon \leq \varepsilon_1$, Eq. (3.1) is uniquely solvable in the sphere $S : \|z\| \leq \delta(\mu)$, and the solution satisfies the inequality

$$\|z\| \leq c\varepsilon^q \tag{3.4}$$

where the constant c is independent of ε.

Proof. Taking, for example, $\mu = 1/(2\|A^{-1}\|)$, one can deduce from inequality (3.2) that for $0 < \varepsilon \leq \varepsilon_0$ the operator $A^{-1}F$ is contractive in the sphere S.

For $z_2 = 0$, $0 < \varepsilon \leq \varepsilon_0$, it follows from (3.2) that

$$\|A^{-1}F(z_1,\varepsilon)\| \leq \|A^{-1}\|\mu\|z_1\| + \|A^{-1}\|\|F(0,\varepsilon)\|$$

Hence, in view of (3.3), it follows that if

$$0 < \varepsilon \leq \varepsilon_1 = \min\left(\varepsilon_0(\mu),\ (\delta(\mu)/\left(2\|A^{-1}\|d\right)\right)^{1/q}$$

then the operator $A^{-1}F$ maps S into itself.

By the Contractive Mapping Principle, Eq. (3.1) with $0 < \varepsilon \leq \varepsilon_1$ has a unique solution in the sphere S, which can be found by the method of successive approximations.

We will show that all successive approximations $z_{i+1} = A^{-1}F(z_i, \varepsilon)$, $0 < \varepsilon \leq \varepsilon_1$, will not exceed $c\varepsilon^q$ in norm (where c is the same constant for all approximations).

As zeroth approximation, we take $z_0 = 0$.

By inequality (3.3), we have

$$\|z_1\| = \|A^{-1}F(0,\varepsilon)\| \leq \|A^{-1}\|d\varepsilon^q$$

Using inequality (3.2) with $\mu = 1/(2\|A^{-1}\|)$, $0 < \varepsilon \leq \varepsilon_0$, we obtain

$$\|z_2\| \leq \|z_2 - z_1\| + \|z_1\| = \|A^{-1}F(z_1,\varepsilon) - A^{-1}F(0,\varepsilon)\| +$$

$$+\|z_1\| \leq \frac{1}{2}\|z_1\| + \|z_1\| \leq 2\|A^{-1}\|d\varepsilon^q$$

We now apply mathematical induction.

Suppose that for $0 \leq j \leq i$, the following inequality is satisfied

$$\|z_j\| \leq 2\|A^{-1}\|d\varepsilon^q$$

We now have

$$\|z_{i+1}\| \leq \|z_{i+1} - z_i\| + \|z_i - z_{i-1}\| + \ldots + \|z_2 - z_1\| + \|z_1\|$$

For $j \leq i$, $\mu = 1/(2\|A^{-1}\|)$ and $0 < \varepsilon \leq \varepsilon_0$, it follows from inequality (3.2) that

$$\|z_{j+1} - z_j\| = \|A^{-1}F(z_j,\varepsilon) - A^{-1}F(z_{j-1},\varepsilon)\| \leq \frac{1}{2}\|z_j - z_{j-1}\| =$$

$$= \frac{1}{2}\|A^{-1}F(z_{j-1},\varepsilon) - A^{-1}F(z_{j-2},\varepsilon)\| \leq \frac{1}{2^2}\|z_{j-1} - z_{j-2}\| \leq \ldots \leq \frac{1}{2^j}\|z_1\|$$

Then

$$\|z_{i+1}\| \leqslant \left(\frac{1}{2^i} + \frac{1}{2^{i-1}} + \ldots + \frac{1}{2} + 1\right)\|z_1\| \leqslant 2\|z_1\| \leqslant c\varepsilon^q$$

where $c = 2\|A^{-1}\|d$ is independent of the order of the approximation and of ε.

Hence, the solution of Eq. (3.1) for $0 < \varepsilon \leqslant \varepsilon_1$ satisfies estimate (3.4).

Given Conditions 1 and 2 and assuming $\varepsilon \neq 0$ to be sufficiently small, we shall now show that problem (1.2), (1.3), (2.8)–(2.9) is uniquely solvable. With due attention to the notation (2.21) for $j = n + 1$, $\varepsilon \neq 0, n \geqslant 1$, we can state that the solvability of this problem is equivalent to the solvability of the following problem

$$\begin{aligned}
&\Delta x(l+1) = \left(\bar{f}_l\right)_x \Delta x(l) + \varepsilon\left(\bar{g}_l\right)_u \Delta u(l) + \Big[f_l\left(\tilde{x}_n(l) + \Delta x(l)\right) + \\
&+ \varepsilon g_l\left(\tilde{x}_n(l) + \Delta x(l),\ \tilde{u}_{n-1}(l) + \Delta u(l)\right) - \tilde{x}_n(l+1) - \left(\bar{f}_l\right)_x \Delta x(l) - \varepsilon\left(\bar{g}_l\right)_u \Delta u(l)\Big], \quad \Delta x(0) = 0 \\
&\varepsilon\Delta\psi(l) = \varepsilon\Delta\psi(l+1)\left(\bar{f}_l\right)_x + \Big[\varepsilon\Big(-\big(F_l\left(\tilde{x}_n(l) + \Delta x(l)\right) + \\
&+ \varepsilon G_l\left(\tilde{x}_n(l) + \Delta x(l),\ \tilde{u}_{n-1}(l) + \Delta u(l)\right)\big)_x + \\
&+ \left(\tilde{\psi}_{n-1}(l+1) + \Delta\psi(l+1)\right)\left(f_l\left(\tilde{x}_n(l) + \Delta x(l)\right) + \varepsilon g_l\left(\tilde{x}_n(l) + \Delta x(l),\ \tilde{u}_{n-1}(l) + \Delta u(l)\right)\right)_x - \\
&- \tilde{\psi}_{n-1}(l) - \Delta\psi(l+1)\left(\bar{f}_l\right)_x\Big)\Big] \\
&\varepsilon\Delta\psi(N) = \Big[-\varepsilon\left(F_N\left(\tilde{x}_n(N) + \Delta x(N)\right)\right)_x - \varepsilon\tilde{\psi}_{n-1}(N)\Big] \\
&\varepsilon\Delta u(l) = \varepsilon R_l^{-1}\Delta\psi(l+1)\left(\bar{g}_l\right)_u + \Big[\varepsilon R_l^{-1}\Big(-G_l\left(\tilde{x}_n(l) + \Delta x(l),\ \tilde{u}_{n-1}(l) + \Delta u(l)\right)\Big)_u + \\
&+ \left(\tilde{\psi}_{n-1}(l+1) + \Delta\psi(l+1)\right)\left(g_l\left(\tilde{x}_n(l) + \Delta x(l),\ \tilde{u}_{n-1}(l) + \Delta u(l)\right)\right)_u + R_l \Delta u(l) - \Delta\psi(l+1)\left(\bar{g}_l\right)_u\Big)\Big]
\end{aligned} \tag{3.5}$$

It is obvious that the corresponding homogeneous linear system

$$\begin{aligned}
&\Delta x(l+1) = \left(\bar{f}_l\right)_x \Delta x(l) + \varepsilon\left(\bar{g}_l\right)_u \Delta u(l), \quad \Delta x(0) = 0 \\
&\varepsilon\Delta\psi(l) = \varepsilon\Delta\psi(l+1)\left(\bar{f}_l\right)_x, \quad \varepsilon\Delta\psi(N) = 0 \\
&\varepsilon\Delta u(l) = \varepsilon R_l^{-1}\Delta\psi(l+1)\left(\bar{g}_l\right)_u
\end{aligned} \tag{3.6}$$

has only a trivial solution.

Let z be the vector in the space $R^{2m(N+1)+rN}$ whose components are $\Delta x(k)$, $\varepsilon\Delta\psi(k)$, $\varepsilon\Delta u(l)$, $m = \dim X, r = \dim U$. Then system (3.5) may be written in the form (3.1), where A is the linear operator defined by the homogeneous linear system (3.6) and which acts in $R^{2m(N+1)+rN}$, while the form of F is obvious if one compares systems (3.5) and (3.6) and is defined by the functions in square brackets in (3.5). The unique solvability of system (3.6) implies that the operator A is invertible.

Using the theorem of finite increments, one can show that, for every function $f(\Delta x(l), \Delta u(l), \Delta\psi(l+1), l, \varepsilon)$ in the bracketed expression in (3.5), for any $\mu > 0$ as small as desired and sufficiently small ε, $\|x_i\|$, $\|u_i\|$, $\|\psi_i\|$ $(i = 1, 2)$, the following inequality holds

$$\left\|f(x_1, u_1, \psi_1, l, \varepsilon) - f(x_2, u_2, \psi_2, l, \varepsilon)\right\| \leqslant \mu\left(\|x_1 - x_2\| + \varepsilon\|u_1 - u_2\| + \varepsilon\|\psi_1 - \psi_2\|\right)$$

By the equations defining the functions $\tilde{x}_n$, $\tilde{u}_{n-1}$, $\tilde{\psi}_{n-1}$, we have

$$\|f(0,0,0,l,\varepsilon)\| = O\left(\varepsilon^{n+1}\right)$$

Thus, Conditions 1–3 of Lemma 1 hold for system (3.5) when written in the form of (3.1). The lemma implies the following.

Lemma 2. Given Conditions 1 and 2 and sufficiently small $\varepsilon > 0$, problem (1.2), (1.3), (2.8)–(2.10) has a unique solution in the neighbourhood of $(\tilde{x}_n, \tilde{u}_{n-1}, \tilde{\psi}_{n-1})$ and the following estimates hold

$$\begin{gathered}\|x(k)-\tilde{x}_n(k)\| = \|\Delta x(k)\| \leqslant c\varepsilon^{n+1} \\ \|u(l)-\tilde{u}_{n-1}(l)\| = \|\Delta u(l)\| \leqslant c\varepsilon^n, \quad \|\psi(k)-\tilde{\psi}_{n-1}(k)\| = \|\Delta\psi(k)\| \leqslant c\varepsilon^n\end{gathered} \tag{3.7}$$

where the constant c is independent of k, l and ε.

Thus, one can construct an asymptotic expansion of the solution of problem (1.2), (1.3), (2.8)–(2.10) in non-negative integral powers of ε, with the remainder terms Δx, Δu, $\Delta\psi$ of the asymptotic expansion satisfying inequalities (3.7).

We now introduce the notation

$$|||x||| = \max_{0\leqslant k\leqslant N}\|x(k)\|, \quad |||u||| = \max_{0\leqslant l\leqslant N-1}\|u(l)\|$$

Let u^* be some fixed control and let x^* be the corresponding trajectory, i.e., the solution of problem (1.2), (1.3) for $u = u^*$.

Lemma 3. For any $r > 0$ constants $\varepsilon_0 > 0$ and $c > 0$ exist such that for $0 < \varepsilon \leqslant \varepsilon_0$, $|||u^* - u||| \leqslant r$

$$|||x^* - x||| \leqslant c\varepsilon|||u^* - u|||$$

where x is the trajectory corresponding to the control u.

The proof of this lemma follows from the form of Eq. (2.3), the continuous differentiability of the functions f_l and g_l and the theorem of finite increments.

Now, assuming that Conditions 1 and 2 are satisfied and that $\varepsilon \neq 0$ is sufficiently small, we shall show that Problem P_ε is uniquely solvable in some neighbourhood of the control u_0.

Let x^*, ψ and u^* denote the solution of problem (1.2), (1.3), (2.8)–(2.10), which exists by Lemma 2.

Lemma 4. Given Conditions 1 and 2 and sufficiently small $\varepsilon > 0$, the function $u^*(l)$ is a locally optimal control for Problem P_ε.

Proof. Together with the trajectory x^* corresponding to control u^*, let us consider the trajectory x (the solution of problem (1.2), (1.3)) corresponding to a control u in some neighbourhood of u^*. By Lemma 3, relations (1.2) yield

$$\begin{gathered}x(l+1)-x^*(l+1) = (f_l+\varepsilon g_l)_x^*\big(x(l)-x^*(l)\big)+\varepsilon(g_l)_u^*\big(u(l)-u^*(l)\big)+ \\ +\frac{1}{2}\varepsilon(g_l)_{u^2}^*\big(u(l)-u^*(l)\big)^2+O\big(\varepsilon^2|||u-u^*|||^2\big)+O\big(\varepsilon|||u-u^*|||^3\big)\end{gathered} \tag{3.8}$$

where the asterisk to the right of the parenthesis means that the derivatives are to be evaluated at $x = x^*$, $u = u^*$.

Now, by Lemma 3, we deduce from (1.1) that

$$\begin{gathered}J(u)-J(u^*) = \sum_k (F_k)_x^*\big(x(k)-x^*(k)\big)+\varepsilon\sum_l\Big((G_l)_x^*\big(x(l)-x^*(l)\big)+ \\ +(G_l)_u^*\big(u(l)-u^*(l)\big)+\frac{1}{2}(G_l)_{u^2}^*\big(u(l)-u^*(l)\big)^2\Big)+O\big(\varepsilon^2|||u-u^*|||^2\big)+O\big(\varepsilon|||u-u^*|||^3\big)\end{gathered}$$

Using the expressions for $(F_N)_x^*$, $(F_l+\varepsilon G_l)_x^*$, $(G_l)_u^*$ obtained from (2.9), (2.8) and (2.10), respectively, as well as expression (3.8), we obtain after reduction the relation

$$J(u)-J(u^*) = \frac{1}{2}\varepsilon\sum_l\Big((G_l)_{u^2}^*-\psi(l+1)(g_l)_{u^2}^*\Big)\big(u(l)-u^*(l)\big)^2+O\big(\varepsilon^2|||u-u^*|||^2\big)+O\big(\varepsilon|||u-u^*|||^3\big)$$

Hence, by virtue of estimates (3.7), we obtain

$$J(u)-J(u^*)=\frac{1}{2}\varepsilon\sum_l R_l\left(u(l)-u^*(l)\right)^2+O\left(\varepsilon^2\|u-u^*\|^2\right)+O\left(\varepsilon\|u-u^*\|^3\right) \tag{3.9}$$

Therefore, for sufficiently small $\varepsilon > 0$, the control u^* is indeed locally optimal for Problem P_ε.

Theorem 3. Given Conditions 1 and 2 and sufficiently small $\varepsilon > 0$, Problem P_ε is uniquely solvable in the neighbourhood of the control u_0 and its solution u^*, x^* satisfies the estimates

$$u^*(l)-\tilde{u}_{n-1}(l)=O\left(\varepsilon^n\right),\quad x^*(k)-\tilde{x}_n(k)=O\left(\varepsilon^{n+1}\right) \tag{3.10}$$

$$J_\varepsilon\left(\tilde{u}_{n-1}\right)-J_\varepsilon\left(u^*\right)=O\left(\varepsilon^{2n+1}\right) \tag{3.11}$$

Proof. The solvability follows from Lemmas 2 and 4. Estimates (3.10) follow from (3.7). Estimate (3.11) is derived from (3.9) by setting $u = \tilde{u}_{n-1}$ and using estimates (3.7).

Relation (3.11) means that the order (or degree) of sub-optimality of the control $\tilde{u}_{n-1}$ in Problem P_ε is $2n + 1$ (for the definition, see, e.g. [14]).

Remarks 3. If we let $\bar{x}$ denote the solution of problem (1.2), (1.3) corresponding to the control $\tilde{u}_{n-1}$, then by Lemma 3 and estimate (3.10), we have

$$x^*(k)-\bar{x}(k)=O\left(\varepsilon^{n+1}\right),\quad \bar{x}(k)-\tilde{x}_n(k)=O\left(\varepsilon^{n+1}\right) \tag{3.12}$$

4. Estimates (3.10) and (3.11), which have been obtained for an approximate solution of the optimal control problem for a discrete weakly controllable system, have the same form as those obtained previously [9] for continuous weakly controllable systems.

5. Replacing u by $\tilde{u}_{n-1}$ and x by $\tilde{x}_n$ in functional (1.1), we obtain a value of the functional which, by (3.10), will differ from the optimum by $O(\varepsilon^{n+1})$, while the estimate given by (3.11) for the difference in the values of the functional is of the order of ε^{2n+1}.

The statement of Theorem 3 and the last remark may be illustrated by the following example.

Example. Consider the problem of minimizing the functional

$$J_\varepsilon(u)=x(2)+\varepsilon\left(u^2(0)+u^2(1)\right)$$

on trajectories of the system

$$x(0)=1,\quad x(1)=\left(x(0)\right)^2+\varepsilon u(0),\quad x(2)=\left(x(1)\right)^2+\varepsilon u(1) \tag{3.13}$$

where $\varepsilon > 0$ is a small parameter.

Using equalities (3.13), we infer that the problem of minimizing the functional $J_\varepsilon(u)$ may be dealt with by treating it as a function of the two variables $u(0)$ and $u(1)$. The solution of the problem is

$$\begin{gathered}u^*(0)=-\frac{1}{1+\varepsilon},\quad u^*(1)=-\frac{1}{2}\\ x^*(0)=1,\quad x^*(1)=\frac{1}{1+\varepsilon},\quad x^*(2)=\frac{1}{(1+\varepsilon)^2}-\frac{\varepsilon}{2}\\ J_\varepsilon\left(u^*\right)=\frac{1}{1+\varepsilon}-\frac{\varepsilon}{4}\end{gathered} \tag{3.14}$$

The solution of the problem

$$P_0: x(1)=\left(x(0)\right)^2,\quad x(2)=\left(x(1)\right)^2,\quad x(0)=1$$

is

$$x_0(0)=1,\quad x_0(1)=1,\quad x_0(2)=1$$

The solution of the problem

$$\begin{gathered}P_1: J_1\left(u_0\right)=x_1(2)+u_0^2(0)+u_0^2(1)\to\min_{u_0}\\ x_1(0)=0,\quad x_1(1)=2x_1(0)+u_0(0),\quad x_1(2)=2x_1(1)+u_0(1)\end{gathered}$$

is

$$
\begin{aligned}
&u_0(0)=-1,\ \ u_0(1)=-\tfrac{1}{2}\\
&x_1(0)=0,\ \ x_1(1)=-1,\ \ x_1(2)=-\tfrac{5}{2}
\end{aligned}
\tag{3.15}
$$

Then (see (2.5))

$$\tilde{x}_1(0)=1,\ \ \tilde{x}_1(1)=1-\varepsilon,\ \ \tilde{x}_1(2)=1-\tfrac{5}{2}\varepsilon \tag{3.16}$$

The solution $\tilde{x}$ of system (3.13) for $u = u_0$,

$$\tilde{x}(0)=1,\ \ \tilde{x}(1)=\left(\tilde{x}(0)\right)^2+\varepsilon u_0(0),\ \ \tilde{x}(2)=\left(\tilde{x}(1)\right)^2+\varepsilon u_0(1)$$

is

$$\tilde{x}(0)=1,\ \ \tilde{x}(1)=1-\varepsilon,\ \ \tilde{x}(2)=1-\tfrac{5}{2}\varepsilon+\varepsilon^2 \tag{3.17}$$

Hence we obtain

$$J_\varepsilon(u_0)=\tilde{x}(2)+\varepsilon\left(\left(u_0(0)\right)^2+\left(u_0(1)\right)^2\right)=1-\tfrac{5}{4}\varepsilon+\varepsilon^2 \tag{3.18}$$

Estimates (3.10)–(3.12) with $n = 1$ follow from relations (3.14)–(3.18).

Substituting u_0 for u and $\tilde{x}_1$ for x in the functional $J_\varepsilon(u)$, we get $1 - \tfrac{5}{4}\varepsilon$, which differs from the optimal value $J_\varepsilon(u^*)$ (see (3,14)) by $O(\varepsilon^2)$.

Theorem 4. Given Conditions 1 and 2 and sufficiently small $\varepsilon > 0$, we have

$$J_\varepsilon(\tilde{u}_i)\leqslant J_\varepsilon(\tilde{u}_{i-1}),\quad i=1,2,\ldots,n-1 \tag{3.19}$$

where

$$\tilde{u}_i(l)=\sum_{j=0}^{i}\varepsilon^j u_j(l)$$

If $u_i \neq 0$, then (3.19) is a strict inequality.

Proof. If $u_i(l) = 0$, inequality (3.19) is obvious.

Let us consider the case when $u_i \neq 0$. Expand the solution of problem (1.2), (1.3) for $u(l) = \tilde{u}_s(l)$ ($s = i - 1, i$) in a series of non-negative integral powers of ε. Then, by the algorithm for determining the terms of expansion (2.1), the solution will have the form

$$\sum_{j=0}^{s+1}\varepsilon^j x_j(k)+O\left(\varepsilon^{s+2}\right)$$

(see the second estimate in (3.12)).

Expanding $J_\varepsilon(\tilde{u}_s)$ ($s = i - 1, i$) in series (2.2) and using Theorem 2, we obtain

$$
\begin{aligned}
J_\varepsilon(\tilde{u}_i)&=\sum_{j=0}^{2i}\varepsilon^j J_j+\varepsilon^{2i+1}\left(\hat{J}_{2i+1}+\tilde{J}_{i+1}(u_i)\right)+O\left(\varepsilon^{2i+2}\right)\\
J_\varepsilon(\tilde{u}_{i-1})&=\sum_{j=0}^{2i}\varepsilon^j J_j+\varepsilon^{2i+1}\left(\hat{J}_{2i+1}+\tilde{J}_{i+1}(0)\right)+O\left(\varepsilon^{2i+2}\right)
\end{aligned}
\tag{3.20}
$$

where $\hat{J}_{2i+1}$ depends on u_j ($j = 0, 1, \ldots, i - 1$), x_j ($j = 0, 1, \ldots, i$).

Since u_i is a solution of the linearly quadratic problem P_{i+1}, which is to minimize the functional $\tilde{J}_{i+1}(u_i)$, it follows, by the uniqueness of the optimal control when $u_i \neq 0$, that

$$\tilde{J}_{i+1}(u_i)<\tilde{J}_{i+1}(0)$$

Hence, using also Eqs (3.2), it follows that inequality (3.19) is true for sufficiently small $\varepsilon > 0$.

We have thus established that the values of the minimized functional do not increase with each new approximation of the control.

The research was supported financially by the Russian Foundation for Basic Research (99-01-00968).

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Translated by D.L.